PHY180 Torque

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1 Torque and "Line of Action":

$$\tau = Fr\perp \text{ where } r\perp = rsin\phi$$
$$\tau = Frsin\phi$$

Relationship between:

$$\tau = rFsin\phi$$

and

$$\overrightarrow{\tau} = \overrightarrow{r} \cdot \overrightarrow{F}$$

What's the cross product of this?:

$$\overrightarrow{\tau} = \overrightarrow{r} \cdot \overrightarrow{F}$$
$$\overrightarrow{\tau} = \begin{bmatrix} rcos\theta \\ rsin\theta \\ 0 \end{bmatrix} x \begin{bmatrix} Fcos\theta_F \\ Fsin\theta_F \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & 0 \\ F_x & F_y & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r_xF_y - r_yFx \end{bmatrix}$$
$$\overrightarrow{\tau} = 0\hat{x} + 0\hat{y} + rcos\theta Fsin\theta_F - rsin\theta Fcos\theta_F$$
$$\overrightarrow{\tau} = rFsin(\theta_f - \theta)\hat{z}$$
$$= rFsin\phi \cdot \hat{Z}$$

$\overrightarrow{r}, \overrightarrow{F}$ in xy plane.
Then we can describe all of rotation as vectors in Z:

$$\overrightarrow{\tau} = \tau_Z\hat{Z}$$
$$\overrightarrow{\omega} = \hat{Z}$$
$$\overrightarrow{\alpha} = \alpha\hat{Z}$$

2 Free Rotation

When unconstrained, a free body will always rotate about it's center of mass. Its motion can be decomposed into two parts, translation of CM, and rotation about CM.

Angular Momentum as cross-product
Relationship between $L = I\omega_0$

Consider an object in circular(rotation) / revolutionary motion.

$$v = r\omega_0$$

$$\overrightarrow{r} = r\hat{r}$$
$$\overrightarrow{L} = \overrightarrow{r} x \overrightarrow{p}$$
$$= r\hat{r} \times mv\hat{t}$$
$$rmv(\hat{r} \times \hat{t}) = \hat{Z}$$
$$= rmv\hat{Z}$$
$$= r^2 m\omega_0\hat{Z}$$
$$I\omega_o\hat{Z}$$
$$\overrightarrow{L} = L_0\hat{Z} = I\omega_0\hat{Z}$$

Example: Torque Caused by Gravity → Angular Acceleration
Consider a pulley attached by a string wrapped around a pulley of radius r, with mass m.
The force on the mass itself will be:

$$F = -mg\hat{y}$$

$$\overrightarrow{\tau_{ext} = I\overrightarrow{\alpha}}$$
$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$

But we know:

$$\overrightarrow{r} = -R\hat{x}$$

and

$$\overrightarrow{F} = -mg\hat{y}$$

Therefore:

$$= -R(-mg)(\hat{x} \times \hat{y})$$

$$= mgR\hat{z}$$

$$\overrightarrow{\tau} = mgR$$

$$I\overrightarrow{\alpha} = mgR\overrightarrow{z}$$

$$\alpha_0 = \frac{mgR}{I}$$

where:

$$\overrightarrow{\alpha} = \alpha_0 \hat{z}$$

At any point in time we have these expressions:

$$\omega_0 = \alpha_0 \Delta t$$

and

$$\Delta\theta = \frac{1}{2}\alpha_0 \Delta t^2$$